MTH 605: Topology I

Homework I

(Due 18/08)

1. Show that the topologies $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not compatible.
2. Show that each of following collections define basis for a topology on X. Describe the topology generated in each case.
   (a) $\mathcal{B} = \{(a, b) \,|\, a < b,\ a \text{ and } b \text{ rational}\}$, $X = \mathbb{R}$.
   (b) $\mathcal{C} = \{[a, b) \,|\, a < b,\ a \text{ and } b \text{ rational}\}$, $X = \mathbb{R}$.
   (c) $\mathcal{D} = \{(a, b) \times (c, d) \,|\, a < b,\ c < d,\ a, b, c \text{ and } d \text{ rational}\}$, $X = \mathbb{R}^2$.
3. If A, B, and A_α are subsets of a space X. Determine whether the following statements hold. Prove them if they are true, and give a counterexample if they are false.
   (a) If $A \subset B$, then $\overline{A} \subset \overline{B}$.
   (b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
   (c) $\overline{\cup A_\alpha} \supset \cup \overline{A_\alpha}$.
   (d) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
   (e) $\overline{\cap A_\alpha} = \cap \overline{A_\alpha}$.
   (f) $\overline{A - B} = \overline{A} - \overline{B}$.
4. Show that X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \,|\, x \in X\}$ is closed in $X \times X$.
5. If $A \subset X$, we define the boundary of A (denoted by ∂A) by $\partial A = \overline{A} \cap \overline{(X - A)}$. Show the following.
   (a) $A^\circ \cap \partial A = \emptyset$ and $\overline{A} = A^\circ \cup \partial A$.
   (b) $\partial A = \emptyset$ if and only is A is both open and closed.
   (c) U is open if and only if $\partial U = \overline{U} - U$.
6. Find the ∂A and A°, if A is one of the following subsets of $\mathbb{R}^2$.
   (a) $A = \mathbb{Q} \times \mathbb{R}$.
   (b) $A = \{(x, y) \,|\, 0 < x^2 - y^2 \leq 1\}$.
   (c) $A = \{(x, y) \,|\, x \neq 0 \text{ and } y = 1/x\}$.